

# Pseudoscalar-exchange contribution to $(g - 2)_\mu$ from rational approximants

Pablo Sanchez-Puertas<sup>1</sup> and Pere Masjuan<sup>2</sup>

PRISMA Cluster of Excellence, Institut für Kernphysik, Johannes Gutenberg-Universität,  
Mainz D-55099, Germany

E-mail: <sup>1</sup>sanchezp@kph.uni-mainz.de, <sup>2</sup>masjuan@kph.uni-mainz.de

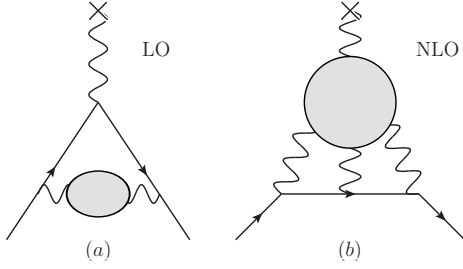
**Abstract.** We present our recent results on the hadronic light-by-light pseudoscalar-exchange contribution to the anomalous magnetic moment of the muon using rational approximants. Our work provides a generalization of Padé approximants to the bivariate case to describe the most general doubly-virtual transition form factor, which is required in the calculation. This method provides a powerful tool to systematically implement the known experimental data on transition form factors in the space-like region and low energies in a model-independent way. Given the lack of experimental data on the doubly-virtual transition form factor, we make use of the pseudoscalar decays into lepton pairs. We find an interesting and puzzling situation which calls for new experimental measurements to clarify the present state.

## 1. Introduction

There has been a persistent discrepancy among the theoretical prediction and the experimental value for the anomalous magnetic moment of the muon,  $a_\mu^{exp} = 116592091(63) \times 10^{-11}$  with  $a_\mu \equiv (g - 2)_\mu$ , at the  $3\sigma$  level [1, 2, 3] which may call for physics beyond the standard model. However, to reach to such conclusion, higher precision is required. For this reason there are two different projected experiments at Fermilab [4] and JPARC [5] aiming for a precision at around  $10^{-10}$ .

Regretfully, the experimental effort will not be enough if it is not complemented by an equally precise theoretical prediction, which uncertainty is totally given by the hadronic contributions [1]. However, these calculations are not an easy task as they involve quantum chromodynamics (QCD) at all energy scales, which represents a multi-scale problem and has required a close collaboration among theorists and experimentalists. A beautiful example is the leading hadronic contribution to  $(g - 2)_\mu$ , the hadronic vacuum polarization (HVP), which is shown in Fig. 1a, where the blob stands for all-possible intermediate hadronic states. There, the optical theorem relates this quantity to experimentally-measurable cross sections  $\sigma(e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons})$ , and an ambitious experimental program promises to achieve the required precision for this quantity, which at present is of  $43 \times 10^{-11}$  [6].

Certainly, more complicated is the situation for the hadronic light-by-light (HLBL) contribution, which appears at NLO together with the NLO corrections to the HVP and is shown in Fig. 1b. This may become the dominant theoretical uncertainty, which at the moment



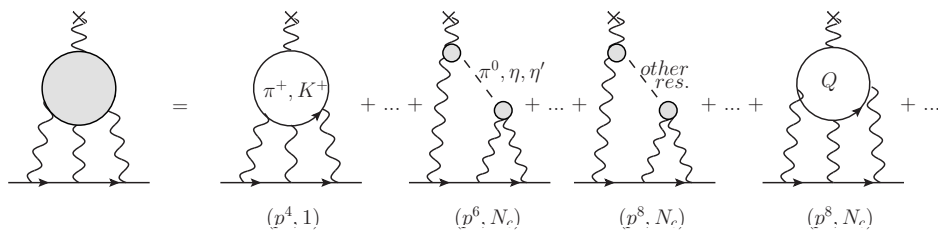
**Figure 1.** Hadronic contributions to  $(g - 2)_\mu$ . Diagram (a) represents the hadronic vacuum polarization, whereas diagram (b) stands for the hadronic light-by-light scattering contribution.

is estimated as  $40 \times 10^{-11}$  [6]. In contrast to the HVP, the HLBL cannot be directly related to any measurable cross section, and again requires the knowledge of QCD at all scales, which is represented in Fig. 1b by the grey blob. Being a multi-scale problem, it is not possible to rely on a perturbative  $\alpha_s$  expansion. However, it is still possible to find guidance on the chiral and large- $N_c$  expansions, which allowed in Ref. [7] to decompose the HLBL in the different contributions depicted in Fig. 2. In this scheme, the  $\pi^0, \eta$  and  $\eta'$  exchanges together with the (numerically subleading)  $\pi^\pm$  and  $K^\pm$  loops give the major contributions, see Tab. 2 in Ref. [3]. This has triggered an ambitious program and great interest in the field of  $\gamma\gamma$  physics. Still, such calculations are complicated as there is not an obvious tool to implement all the available data and theoretical constraints. Often, one relies then on simplified models where the intrinsic errors are difficult to estimate. The current reference numbers,  $116(39) \times 10^{-11}$  [1] and  $105(25) \times 10^{-11}$  [8], suffer indeed from this problem. In order to supply this shortcoming, different approaches have been proposed. As an example, lattice calculations [9, 10], Dyson-Schwinger equations [11] and dispersive approaches [12, 13, 14] have been proposed. The last are based on the use of data at low energies, but lack the ability to implement the high-energy constraints. In our study, we propose a mathematical framework which allows to make full use of data and high-energy constraints and describe, in a model-independent fashion, the required quantities for calculating the  $\pi^0, \eta$  and  $\eta'$  contribution to the HLBL, which represents the dominant piece. For this, we extend the approach based on Padé approximants described in Refs. [15, 16] to the double virtual case. For the moment, we have to face the situation that no double-virtual data is available. At this respect, we discuss in the last section what can be learnt from  $P \rightarrow \ell\ell$  decays, where  $P = \pi^0, \eta, \eta'$ .

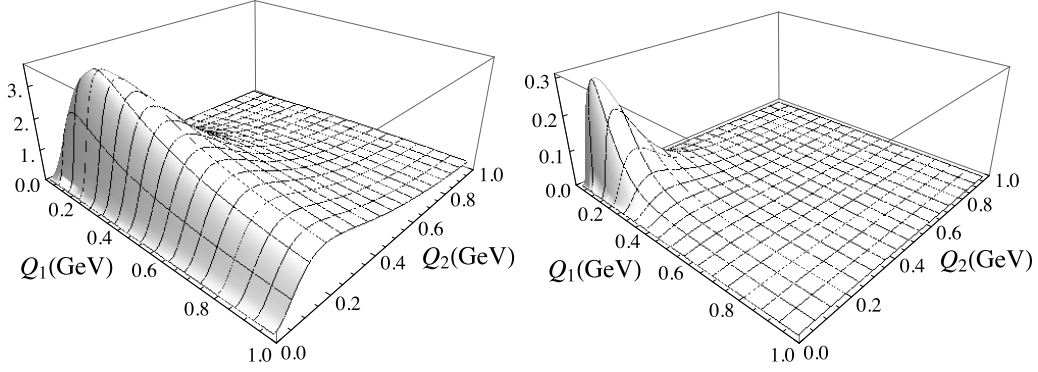
## 2. Pseudoscalar-exchange hadronic light-by-light contribution to $(g - 2)_\mu$

Given that the (light) pseudoscalar exchange represents the major contribution to the HLBL, such calculation becomes of central importance when aiming for precision. This requires an accurate description of  $P\gamma^*\gamma^*$  interactions, which are represented as the grey blobs in Fig. 2, and are described in terms of the pseudoscalar transition form factor (TFF)  $F_{P\gamma^*\gamma^*}(q_1^2, q_2^2)$  as

$$i\mathcal{M} = i\epsilon_{\mu\nu\rho\sigma}\epsilon_1^\mu q_1^\nu \epsilon_2^\rho q_2^\sigma F_{P\gamma^*\gamma^*}(q_1^2, q_2^2), \quad (1)$$



**Figure 2.** The combined chiral and large- $N_c$  counting for HLBL proposed in Ref. [7].



**Figure 3.** Integrals from Eq. (2) for a constant TFF and  $t = 0$  (the  $(\alpha/\pi)^3$  factor is omitted). Left and right graphics represent the first and second term in the integral, respectively.

meaning that the  $\pi^0$  (as well as the  $\eta$  and  $\eta'$ ) TFF  $F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$  will play an essential role in determining this quantity. Actually, the contribution to  $a_\mu$  can be expressed in terms of this as [1]

$$a_\mu^{HLBL;P} = \frac{-2\pi}{3} \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dQ_1 dQ_2 \int_{-1}^{+1} dt \sqrt{1-t^2} Q_1^3 Q_2^3 \left[ \frac{F_1 I_1(Q_1, Q_2, t)}{Q_2^2 + m_P^2} + \frac{F_2 I_2(Q_1, Q_2, t)}{Q_3^2 + m_P^2} \right], \quad (2)$$

where the  $I_i(Q_1, Q_2, t)$  functions may be found in Ref. [1],  $Q_3^2 = Q_1^2 + Q_2^2 + Q_1 Q_2 t$ , and the TFF appears through the quantities

$$F_1 = F_{P\gamma^*\gamma^*}(-Q_1^2, -Q_3^2) F_{P\gamma^*\gamma}(-Q_2^2, 0), \quad F_2 = F_{P\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) F_{P\gamma^*\gamma}(-Q_3^2, 0), \quad (3)$$

with  $Q_i^2$  a space-like variable and  $t$  an angular one. The behavior for this integral is shown in Fig. 3 for  $t = 0$ , though very similar shape is obtained for other values. A relevant observation is that the previous integral is peaked at very low space-like energies. Aiming for very small errors, smaller than 10%, it is extremely important to provide a precise description at these scales around (0 – 1) GeV. In addition, specially for the first term in the integral in Eq. (2) which is UV-divergent for a point-like TFF, it is important to provide the adequate high-energy behavior dictated by perturbative QCD—which is the drawback in dispersion relation approaches. This makes a clear statement that, for reaching a precise determination for this quantity, we need to provide a precise description of the pseudoscalars TFFs at very low-energies together with the appropriate high-energy behavior, which are the main concerns of our approach discussed below.

### 3. A rational description for $F_{P\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$

Achieving a TFF description from first principles is extremely challenging, if not impossible, at the moment. On the one hand, at the very low energies involved in the process, one could think about resorting to a chiral perturbation theory ( $\chi$ PT) approach. However, any description beyond a constant TFF requires many unknown low-energy constants and violates unitarity at high energies. Moreover, the convergency of this expansion breaks down much before the first resonance scale, still relevant in our integral. For details on a  $\chi$ PT-based calculation for the HLBL, see Ref. [17]. On the other hand, for very large energies, we can rely on perturbative QCD. In this framework, the TFF is obtained in terms of a convolution of a hard-scattering amplitude with the pseudoscalar distribution amplitude [18]. The latter object, which is of non-perturbative nature, encodes the  $\pi^0$  structure and must be modeled, inducing potential large

uncertainties. Consequently, the only reliable limits at disposal are [18, 19, 20]

$$\lim_{Q^2 \rightarrow \infty} Q^2 F_{P\gamma^*\gamma}(-Q^2, 0) = 2F_\pi, \quad \lim_{Q^2 \rightarrow \infty} Q^2 F_{P\gamma^*\gamma^*}(-Q^2, -Q^2) = (2/3)F_\pi, \quad (4)$$

where  $F_\pi = 92$  MeV is the pion decay constant [2]. Therefore, it is necessary to find an alternative approach able to link both, the low and the high energies as well as controlling the theoretical uncertainties. For this, some models, such as hidden local symmetry [21], resonant chiral theory [22], or large- $N_c$  based generalizing vector meson dominance ideas [23, 24] have been proposed. Their qualitative agreement may be understood from the large- $N_c$  limit of QCD, where Green's functions are expressed in terms of an infinite number of resonances exchange [25, 26]. They lack however the estimation of the theoretical error which is associated to their model or simplifying assumptions, and the deeper question remains on whether it would be possible in these models to reproduce the TFF to arbitrary precision. It was proposed however in Ref. [27], that the agreement of this approaches can be understood as well from the framework of Padé theory. The advantage of the latter is that, for certain kind of functions, it allows to go beyond the large- $N_c$  limit, and approximate the real world QCD function. Actually, these properties have already been exploited for dealing with the HVP [28, 29] and it has been shown that they allow to estimate the theoretical uncertainties. We propose to use this framework for reproducing the TFF arising in the pseudoscalar exchange contribution to HLBL as well [15, 16].

Given an analytic function with known series expansion, say,  $F_{P\gamma^*\gamma}(-Q^2, 0) = F_{P\gamma\gamma}(0, 0)(1 - b_P Q^2 + c_P Q^4 + \dots)$ , Padé approximants (PA) are rational functions of two polynomials,  $R_N(Q^2)$  and  $Q_M(Q^2)$  of degree  $N$  and  $M$ , respectively, constructed such as to match the original series expansion up to order  $\mathcal{O}((Q^2)^{N+M+1})$  terms [30],

$$P_M^N(Q^2) = \frac{R_N(Q^2)}{Q_M(Q^2)} = F_{P\gamma\gamma}(0, 0)[1 - b_P Q^2 + c_P Q^4 + \dots + \mathcal{O}((Q^2)^{N+M+1})]. \quad (5)$$

Padé approximants can be proven to approximate not only meromorphic functions —which represents the large- $N_c$  limit of QCD— but Stieltjes functions as well [30]. These kind of functions may have threshold discontinuities and often represents cases of physical interest [28]. Padé theory guarantees then the convergence of some kind of sequences  $P_{N+M}^N(Q^2)$  in the whole complex-plane except for the discontinuity itself, where the original function is ill-defined. These theorems provide then a mathematical corpus of the model-independedency of our approach as the known analytic structure of TFF seems to fall in this kind of functions [31]. In this respect, we are going a step further with respect to previous approaches, guaranteeing the ability to reproduce, to arbitrary precision, the TFF. Note that this holds for some previous approaches in the strict large- $N_c$  limit of QCD alone [27]. Moreover, the systematic construction, Eq. (5) allows to estimate the theoretical uncertainty [15, 32, 28, 16, 31].

Still, it is necessary to provide the low-energy expansion, Eq. (5), before we can apply the method. In this respect, it was shown in Ref. [32], that PAs can be used as well for this purpose in order to safely extract the low-energy parameters from experimental data for the space-like and low-energy time-like TFF. In Refs. [28, 16, 31], we were able to extract in this way some of the leading parameters for the  $\pi^0$ ,  $\eta$  and  $\eta'$ , allowing us to reconstruct the first approximants and obtain valuable information, as the  $\eta$  and  $\eta'$  mixing parameters. This allows to safely reconstruct the TFFs at that low-energies where often no data is available.

So far, this provides a description for the single virtual TFF  $F_{P\gamma^*\gamma}(Q^2, 0)$ . However, from Eqs. (2) and (3), we find that it is the most general doubly-virtual TFF  $F_{P\gamma^*\gamma^*}(Q^2, Q^2)$  that we require in our calculation. To describe this, we need to extend the PAs to the bivariate case

**Table 1.**  $a_\mu^{HLBL;\pi^0\eta,\eta'}$  preliminary results for different values in the chosen  $a_{1,1}$  range.

Units of $10^{-10}$	$\pi^0$	$\eta$	$\eta'$	Total
$a_{1,1} = 2b_P^2$ [OPE]	6.64(33)	1.69(6)	1.61(21)	9.94(40) <sub>stat</sub> (50) <sub>sys</sub>
$a_{1,1} = b_P^2$ [Fact]	5.53(27)	1.30(5)	1.21(12)	8.04(30) <sub>stat</sub> (40) <sub>sys</sub>
$a_{1,1} = 0$	5.10(23)	1.16(7)	1.07(15)	7.33(28) <sub>stat</sub> (37) <sub>sys</sub>

for symmetric functions, which are known as Canterbury approximants (CA), see Ref. [33] and references therein. They are constructed, similarly to PA, from the original series expansion and the simplest approximant reads [33]

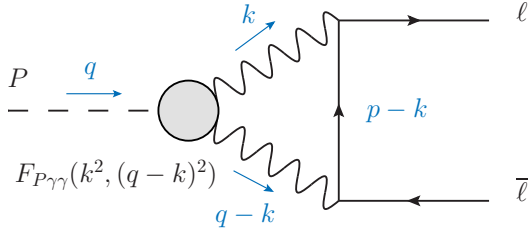
$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0, 0)}{1 + b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{1,1})Q_1^2Q_2^2}, \quad (6)$$

with  $b_P$  the TFF slope and  $a_{1,1}$  given by the doubly-virtual expansion  $F_{P\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) = F_{P\gamma\gamma}(0, 0)(1 - b_P(Q_1^2 + Q_2^2) + a_{1,1}Q_1^2Q_2^2 + \dots)$ . As already discussed in the introduction, there is at present not available experimental data for the doubly-virtual TFF, which would allow to extract the low-energy parameters in a similar manner as for the single-virtual TFF. This means that we cannot go to large approximants. Then, we stick for the moment to the simplest element Eq. (6) and judge, based on theoretical constraints, a reasonable range in which the real  $a_{1,1}$  parameter would lie be contained. On the one side, at low energies,  $\chi$ PT offers indications that the TFF should factorize [34], i.e.,  $F_{P\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \sim F_{P\gamma^*\gamma}(-Q_1^2, 0) \times F_{P\gamma^*\gamma}(0, -Q_2^2)$ , implying then  $a_{1,1} = b_P^2$ . On the other side, at large energies, the OPE condition, Eq. (4), suggest that  $a_{1,1} = 2b_P^2$  so that the high-energy behavior is fulfilled. Guided by this hints, we choose an even more conservative range  $a_{1,1} \in \{0, 2b_P^2\}$  [33], which we believe is generous enough to consider possible departure from factorization at higher energies<sup>1</sup>. Of course, ultimately, experimental data will have the last word on this. Let us remark though, that either of the above-mentioned choices have small impact at low energies —see discussion from the authors in the proceedings of Chiral Dynamics 2015.

#### 4. Results for $a_\mu^{HLBL;\pi^0\eta,\eta'}$

With the doubly-virtual behavior incorporated, we can come back to the HLBL contribution. Using our chosen band for the double virtual parameter  $a_{1,1}$ , which we stress, not only considers the correct low-energy behavior, but account as well for the high-energy one (i.e., it reproduces for  $a_{1,1} = 2b_P^2$  the power-like behavior in Eq. (4)), we obtain the preliminary results for their contribution to HLBL,  $a_\mu^{HLBL;\pi^0\eta,\eta'}$ , in Table 1. There, we quote the statistical error from the data-based procedure together with our estimated 5% systematic error [16]. For the first time, a fully data-driven with high-energy constraints implementation and systematic errors has been provided. We remark that OPE behavior was not considered for the  $\eta$  and  $\eta'$  cases in the reference numbers from Refs. [1, 8] which used a factorization approach based on resonance ideas which are not required in our framework. We find that the error on this assumption may not be negligible, i.e. at the order of the projected experimental precision  $\sim 16 \times 10^{-11}$ . The overall uncertainty is dominated by the  $\pi^0$  contribution, whereas the  $\eta$  result has been greatly improved

<sup>1</sup> The factorization assumption at low energies used in Ref. [33] for estimating our  $a_{1,1}$  range was confirmed in recent dispersive analysis for the  $\eta$  case [35]



**Figure 4.** The leading order QED contribution to the  $P \rightarrow \bar{\ell}\ell$  process where  $P = \pi^0, \eta, \eta'$ .  $F_{P\gamma\gamma}(k^2, (q-k)^2)$  stands for the transition form factor.

as compared to Ref. [16]. This has been possible thanks to the new low-energy parameters extraction in Ref. [31] including, for the first time, not only the space-like, but the time-like data from the precise Dalitz decay measurements from A2 [36] collaboration at Mainz. This illustrates the potentiality of the method to benefit from very different experimental regimes. In principle, new precise data from the  $\pi^0$  TFF would allow for a similar improvement. In this respect, we are currently working on the impact of new data on these errors [37]. Accounting for our quoted range we obtain that [37]

$$a_\mu^{HLBL; \pi^0 \eta, \eta'} = (73.3 \div 99.4)(\pm 6.4) \times 10^{-11}, \quad (7)$$

where the lower(upper) values come from  $a_{1,1} = 0(2b_P^2)$  choice. We stress that, at this point, it is not possible to choose either factorization or OPE as this would be in contradiction either with the high- or low-energy behavior. This is a peculiarity of the lowest approximation and calls for the use of the  $C_2^1(Q_1^2, Q_2^2)$  approximant, which already has the ability to implement both behaviors [37]. We remark that we do not consider the so called off-shell effects [1, 19] in our calculation. See Ref. [38] in this respect. So far, this represents our best estimate, which global uncertainty is clearly dominated by the chosen  $a_{P,1,1}$  window, arising from our ignorance about the doubly-virtual TFF. This is greater than our uncertainty estimation,  $\pm 6.3 \times 10^{-11}$ , and twice as big as the projected experimental  $(g-2)_\mu$  uncertainties  $\sim 16 \times 10^{-11}$ . This clearly illustrates the necessity of measuring the doubly-virtual TFF if we aim to achieve a similar precision as projected experiments. This should answer the question on how fast the TFF reaches the (doubly-virtual) high-energy behavior and would allow for the  $C_2^1(Q_1^2, Q_2^2)$  determination. Given the lack of knowledge both from the experimental side, where no data is available, and the theoretical side, which offer no answer yet, we consider in the next section the  $P \rightarrow \bar{\ell}\ell$  decays, which as we argue, offer an indirect probe to this question.

## 5. Pseudoscalar decays into a lepton pair and $(g-2)_\mu$ implications

Whereas experimentally it is very difficult to probe the doubly-virtual TFF, limited by the low cross sections together with current achieved luminosities, there is an alternative indirect path to probe this. This possibility is brought, in analogy to new physics in  $(g-2)_\mu$ , through loop mediated processes which, being sensible to the TFF at the whole energy scale, provide us the opportunity to test high-energy effects in low-energy phenomena. In this case such possibility is brought by the pseudoscalar decays into lepton pairs,  $P \rightarrow \bar{\ell}\ell$ , for which  $P = \pi^0, \eta, \eta'$ . This process appears at leading order through the diagram shown in Fig. 4. As such, the TFF must be integrated at all energies offering the desired indirect probe. The branching ratio (BR) for this decay may be expressed in terms of the two photon decay width as

$$\frac{\text{BR}(P \rightarrow \bar{\ell}\ell)}{\text{BR}(P \rightarrow \gamma\gamma)} = 2 \left( \frac{\alpha m_\ell}{\pi m_P} \right)^2 \beta_\ell \left| \mathcal{A}(m_P^2) \right|^2, \quad (8)$$

where  $\mathcal{A}(q^2)$  represents the loop amplitude (see Ref. [33] and references therein for details)

$$\mathcal{A}(q^2) = \frac{2i}{\pi^2 q^2} \int d^4k \frac{q^2 k^2 - (q \cdot k)^2}{k^2 (q-k)^2 ((p-k)^2 - m_\ell^2)} \tilde{F}_{P\gamma^*\gamma^*}(k^2, (q-k)^2), \quad (9)$$

which is unknown as far as the normalized TFF ( $\tilde{F}_{P\gamma\gamma}(0,0) = 1$ ) is unspecified. The role of the doubly virtual TFF is actually rather important as the given integral, similarly to the HLBL case, is UV-divergent. Remarkably, for the  $\pi^0$  case, it is possible to go further without a single clue on the TFF. Being the lightest hadron, it is not possible to find any additional intermediate hadronic state which may be on-shell, and contribute therefore to the imaginary part. Consequently, its imaginary part is solely given by the intermediate  $\gamma\gamma$  state, which gives

$$\text{Im}(\mathcal{A}(m_{\pi^0}^2)) = \frac{\pi}{2\beta_\ell} \ln \left( \frac{1 - \beta_\ell}{1 + \beta_\ell} \right) \quad (\beta_\ell = \sqrt{1 - 4m_\ell^2/m_{\pi^0}^2}), \quad (10)$$

inducing the so called unitary bound [39] for the BR,  $|\mathcal{A}(m_P^2)| \geq \text{Im}(\mathcal{A}(m_{\pi^0}^2))$ , which for the  $\pi^0$  gives  $\text{BR}(\pi^0 \rightarrow e^+e^-) = 4.7 \times 10^{-8}$ . This provides a model-independent lower bound, which violation would certainly imply physics beyond the standard model.

Experimentally it is found that, after removing the radiative corrections [40, 41],

$$\text{BR}^{KTeV}(\pi^0 \rightarrow e^+e^-) = 7.48(38) \times 10^{-8} > \text{BR}^{th}(\pi^0 \rightarrow e^+e^-) = 6.23(09) \times 10^{-8}, \quad (11)$$

which is certainly (quite) above the unitary bound. Still, latest results on RC [42, 43] suggest  $\text{BR}^{KTeV}(\pi^0 \rightarrow e^+e^-) = 6.87(38) \times 10^{-8}$ . Nevertheless, they cannot yet justify such deviation, which would still imply some kind of new physics and has attracted much attention [44, 45].

Often, the unitary bound has been extended to the  $\eta$  and  $\eta'$  cases in order to provide an estimate for experimental searches. A word of caution comes here. While for the  $\pi^0$  this was a firm result, this is not the case for the heavier  $\eta$  and  $\eta'$  mesons, which masses allow to include hadronic intermediate states such as  $\pi^+\pi^-\gamma$  or even  $\omega\gamma$  for the heavy  $\eta'$ . We find small corrections for the  $\eta$  to the unitary bound, but large corrections to the  $\eta'$  as the imaginary part is reduced by 20% (see Ref. [46] and our proceedings in Chiral Dynamics 2015).

To go further and provide an estimation for these processes, we need to input some description for the TFF. Then, checking with the experimental values, we can find whether our description, assuming that new physics do not play any role here, corresponds to the experimental world. If this turns out not to be the case, it is possible that our understanding of the doubly-virtual TFF is not as good as we believed. Once more, the crucial observation in this process is the dominance of the low-energy space-like  $Q_1^2 \simeq Q_2^2$  region which fully dominates the integral, see Ref. [33]. This process is perfectly suited then for our CA description. Taking our quoted  $a_{P;1,1}$  range, we obtain the preliminary results in Table 2. We provide the combination of both, statistical and systematic errors. For details on these, see Refs. [33, 46]. We find that, for the  $\pi^0$ , previous results have probably underestimated their theoretical errors when modeling the doubly-virtual TFF [41]. Actually, this feature applies for the  $\eta$  and  $\eta'$  cases as well [41], where not only the double virtuality, but the single-TFF have been crudely modeled [51]. Moreover, we find for the  $\eta$  and  $\eta'$  remarkable deviations with respect to previous results [41, 52] arising from the approximations adopted in previous calculations [33, 51], which cannot be neglected for the  $\eta$  and  $\eta'$  decays. Finally, it is worth reminding that integral (9) is sensible as well to the time-like region up to the  $m_P$  mass. This feature is of special importance for the  $\eta'$ . It can be shown that our method is able to effectively reproduce threshold effects there and we are safe

**Table 2.** Our preliminary results for  $\text{BR}(P \rightarrow \bar{\ell}\ell)$  for  $a_{1,1} = (b_P^2, 2b_P^2)$  range.

Process	$\text{BR}^{th}$	$\text{BR}^{exp}$	Ref.
$\pi^0 \rightarrow e^+e^-$	$(6.20 \div 6.35)(5) \times 10^{-8}$	$7.48(38) \times 10^{-8}$	[40]
$\eta \rightarrow e^+e^-$	$(5.31 \div 5.44)(\begin{smallmatrix} +4 \\ -5 \end{smallmatrix}) \times 10^{-9}$	$\leq 2.3 \times 10^{-6}$	[47]
$\eta \rightarrow \mu^+\mu^-$	$(4.72 \div 4.52)(\begin{smallmatrix} +4 \\ -8 \end{smallmatrix}) \times 10^{-6}$	$5.8(8) \times 10^{-6}$	[48]
$\eta' \rightarrow e^+e^-$	$(1.82 \div 1.86)(19) \times 10^{-10}$	$\leq 5.6 \times 10^{-9}$	[49, 50]
$\eta' \rightarrow \mu^+\mu^-$	$(1.36 \div 1.49)(33) \times 10^{-7}$	—	—

to perform such calculation [46], which is an important and distinctive feature in our approach not implemented so far.

We find from our results that both the  $\pi^0 \rightarrow e^+e^-$  and the  $\eta \rightarrow \mu^+\mu^-$  decays show deviations from their experimental values, at  $2.9\sigma$  and  $1.3\sigma$ , respectively. Remarkably, the latest bounds on  $\eta' \rightarrow e^+e^-$  [49, 50] are reaching our predictions —see the talks at this conference. Therefore, we encourage our experimental colleagues in Novosibirsk to further pursue this decay. Note that including radiative corrections alleviate the first result, leading to  $1.5\sigma$ . Yet its statistical significance, that represents a potential large deviation and, for the moment, the only clue about the doubly-virtual TFF effects. Given that they are dominated by the low-energy region, such effect would be encoded in the lowest-energy double virtual parameter  $a_{1,1}$ . Therefore, we can set it free and match to the experimental values. For the  $\pi^0$ , it requires a large negative  $a_{1,1}$  value, greatly damping the TFF at low energies. In contrast, for the  $\eta$  case this requires an extremely softly-falling TFF, which would require the use of the  $C_2^1(Q_1^2, Q_2^2)$  approximant. Considering both results represents then a puzzling situation. Using this input to calculate the  $a_\mu^{HLBL;\pi^0}$  contribution, we obtain without(with) the latest RC results

$$a_\mu^{HLBL;\pi^0} = 1.3(2.8) \times 10^{-10}. \quad (12)$$

This represents a large shift with respect to our previous prediction in Table 1, much beyond our quoted error. We conclude therefore that such situation calls for an urgent revision. First, some experimental data would be required in order to improve our knowledge of the TFF. Second, a new precise measurement, with the latest results on radiative corrections accounted for, should be pursued. Only this would clarify the current situation and conclude whether we have the situation under control or, perhaps, new physics are playing some role in these decays.

## 6. Conclusions and Outlook

We have presented a model-independent approach based on rational approximants, namely, Canterbury approximants, to describe the pseudoscalar TFFs. This method provides both an useful tool to extract valuable information about TFF from experimental data as well as to reconstruct the TFF itself for later calculations. We have used this model to calculate then the  $\pi^0, \eta$  and  $\eta'$  HLBL contribution to  $(g-2)_\mu$ . We have shown that the current limitation comes from our uncertainty on the double-virtual TFF, for which no data is available yet. To supply this situation, we have made use of  $P \rightarrow \bar{\ell}\ell$  decays. We have found that current experimental values present a puzzling situation which challenge our TFF description. Nevertheless, experimental work is required before any claim, which is specially urgent given the projected accuracy of future  $(g-2)_\mu$  experiments. Meanwhile, to improve our description, we are studying the next



$C_2^1$  approximant, which would allow to implement both, the low- and high-energy constraints at once and, we hope, will provide a better understading of the situation.

## 7. Acknowledgments

We thank Marc Vanderhaeghen for encouragements and discussions. Besides, P. Sanchez-Puertas would like to thank the organizers for their hospitality and the opportunity to participate in this workshop and enjoy from the inspiring environment there. Work supported by the Deutsche Forschungsgemeinschaft DFG through the Collaborative Research Center “The Low-Energy Frontier of the Standard Model” (SFB 1044), eprint: MITP/15-090.

## References

- [1] Jegerlehner F and Nyffeler A 2009 *Phys. Rept.* **477** 1–110 (*Preprint* 0902.3360)
- [2] Olive K A *et al.* (Particle Data Group) 2014 *Chin. Phys.* **C38** 090001
- [3] Masjuan P 2015 *Nucl. Part. Phys. Proc.* **260** 111–115 (*Preprint* 1411.6397)
- [4] Lee Roberts B (Fermilab P989) 2011 *Nucl. Phys. Proc. Suppl.* **218** 237–241
- [5] Mibe T (J-PARC g-2) 2010 *Chin. Phys.* **C34** 745–748
- [6] Knecht M 2015 *Nucl. Part. Phys. Proc.* **258-259** 235–240 (*Preprint* 1412.1228)
- [7] de Rafael E 1994 *Phys. Lett.* **B322** 239–246 (*Preprint* hep-ph/9311316)
- [8] Prades J, de Rafael E and Vainshtein A 2009 *Adv. Ser. Direct. High Energy Phys.* **20** 303–317 (*Preprint* 0901.0306)
- [9] Blum T, Chowdhury S, Hayakawa M and Izubuchi T 2015 *Phys. Rev. Lett.* **114** 012001 (*Preprint* 1407.2923)
- [10] Green J, Gryniuk O, von Hippel G, Meyer H B and Pascalutsa V 2015 (*Preprint* 1507.01577)
- [11] Eichmann G, Fischer C S, Heupel W and Williams R 2014 *11th Conference on Quark Confinement and the Hadron Spectrum (Confinement XI) St. Petersburg, Russia, September 8-12, 2014* (*Preprint* 1411.7876) URL <https://inspirehep.net/record/1331422/files/arXiv:1411.7876.pdf>
- [12] Colangelo G, Hoferichter M, Procura M and Stoffer P 2014 *JHEP* **09** 091 (*Preprint* 1402.7081)
- [13] Colangelo G, Hoferichter M, Kubis B, Procura M and Stoffer P 2014 *Phys. Lett.* **B738** 6–12 (*Preprint* 1408.2517)
- [14] Pauk V and Vanderhaeghen M 2014 *Phys. Rev.* **D90** 113012 (*Preprint* 1409.0819)
- [15] Masjuan P 2012 *Phys.Rev.* **D86** 094021 (*Preprint* 1206.2549)
- [16] Escribano R, Masjuan P and Sanchez-Puertas P 2014 *Phys.Rev.* **D89** 034014 (*Preprint* 1307.2061)
- [17] Ramsey-Musolf M J and Wise M B 2002 *Phys. Rev. Lett.* **89** 041601 (*Preprint* hep-ph/0201297)
- [18] Lepage G P and Brodsky S J 1980 *Phys. Rev.* **D22** 2157
- [19] Melnikov K and Vainshtein A 2004 *Phys. Rev.* **D70** 113006 (*Preprint* hep-ph/0312226)
- [20] Knecht M and Nyffeler A 2001 *Eur. Phys. J.* **C21** 659–678 (*Preprint* hep-ph/0106034)
- [21] Benayoun M, David P, DelBuono L and Jegerlehner F 2015 (*Preprint* 1507.02943)
- [22] Roig P, Guevara A and López Castro G 2014 *Phys. Rev.* **D89** 073016 (*Preprint* 1401.4099)
- [23] Knecht M and Nyffeler A 2002 *Phys. Rev.* **D65** 073034 (*Preprint* hep-ph/0111058)
- [24] Husek T and Leupold S 2015 (*Preprint* 1507.00478)
- [25] ’t Hooft G 1974 *Nucl. Phys.* **B72** 461
- [26] Witten E 1979 *Nucl. Phys.* **B160** 57
- [27] Masjuan P and Peris S 2007 *JHEP* **05** 040 (*Preprint* 0704.1247)
- [28] Masjuan P and Peris S 2010 *Phys. Lett.* **B686** 307–312 (*Preprint* 0903.0294)
- [29] Aubin C, Blum T, Golterman M and Peris S 2012 *Phys. Rev.* **D86** 054509 (*Preprint* 1205.3695)
- [30] Baker G A and Graves-Morris P 1996 *Padé Approximants* 2nd ed (*Encyclopedia of Mathematics and its Applications* no 59) (New York: Cambridge University Press)
- [31] Escribano R, Masjuan P and Sanchez-Puertas P 2015 *Eur. Phys. J.* **C75** 414 (*Preprint* 1504.07742)
- [32] Masjuan P, Peris S and Sanz-Cillero J J 2008 *Phys. Rev.* **D78** 074028 (*Preprint* 0807.4893)
- [33] Masjuan P and Sanchez-Puertas P 2015 (*Preprint* 1504.07001)
- [34] Bijmans J, Kampf K and Lanz S 2012 *Nucl. Phys.* **B860** 245–266 (*Preprint* 1201.2608)
- [35] Xiao C W, Dato T, Hanhart C, Kubis B, Meißner U G and Wirzba A 2015 (*Preprint* 1509.02194)
- [36] Aguar-Bartolome P *et al.* (A2) 2014 *Phys. Rev.* **C89** 044608 (*Preprint* 1309.5648)
- [37] Masjuan P and Sanchez-Puertas P In preparation
- [38] Masjuan P and Vanderhaeghen M 2012 (*Preprint* 1212.0357)
- [39] Drell S 1959 *Nuovo Cim.* **11** 693
- [40] Abouzaid E *et al.* (KTeV Collaboration) 2007 *Phys.Rev.* **D75** 012004 (*Preprint* hep-ex/0610072)
- [41] Dorokhov A E and Ivanov M A 2007 *Phys. Rev.* **D75** 114007 (*Preprint* 0704.3498)

- [42] Vasko P and Novotny J 2011 *JHEP* **1110** 122 (*Preprint* 1106.5956)
- [43] Husek T, Kampf K and Novotny J 2014 *Eur.Phys.J.* **C74** 3010 (*Preprint* 1405.6927)
- [44] Kahn Y, Schmitt M and Tait T M 2008 *Phys.Rev.* **D78** 115002 (*Preprint* 0712.0007)
- [45] Chang Q and Yang Y D 2009 *Phys.Lett.* **B676** 88–93 (*Preprint* 0808.2933)
- [46] Masjuan P and Sanchez-Puertas P In preparation
- [47] Agakishiev G *et al.* (HADES) 2014 *Phys.Lett.* **B731** 265–271 (*Preprint* 1311.0216)
- [48] Abegg R, Baldisseri A, Boudard A, Briscoe W, Fabbro B *et al.* 1994 *Phys.Rev.* **D50** 92–103
- [49] Achasov M N *et al.* (SND) 2015 *Phys. Rev.* **D91** 092010 (*Preprint* 1504.01245)
- [50] Akhmetshin R *et al.* (CMD-3) 2015 *Phys.Lett.* **B740** 273–277 (*Preprint* 1409.1664)
- [51] Dorokhov A, Ivanov M and Kovalenko S 2009 *Phys.Lett.* **B677** 145–149 (*Preprint* 0903.4249)
- [52] Knecht M, Peris S, Perrottet M and de Rafael E 1999 *Phys.Rev.Lett.* **83** 5230–5233 (*Preprint* hep-ph/9908283)